

Electromagnetic radiation of the travelling spin wave propagating in an antiferromagnetic plate. Exact solution.

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Abstract

The exact solution of radiation problem of a spin wave travelling in an antiferromagnetic (AFM) plate was found. The spin wave with in-plane oscillations of antiferromagnetism vector was considered. In this case the magnetization vector is oscillating being perpendicular to the AFM plate and depends on time and plane coordinates as travelling wave does. This model allows to obtain exact analytical expression for Hertzian vector and, consequently, the retarded potentials and field strengths as well.

It is shown that expressions obtained describe Cherenkov radiation caused by the travelling wave. The radiated electromagnetic wave is the *TEM* type if a phase velocity exceeds the speed of light. Otherwise electric and magnetic field values exponentially decrease in the direction normal to the plate. The energy losses were evaluated also.

1 Introduction

It is known that not only particles can be the Cherenkov radiation sources but also, so-called, superlight "reflections" [1] formed by the motion of particles number large enough.

We want to point out that such effects can be observed at spin wave propagation on the magnetic surface. In particular, it is possible to receive

the exact solution of radiation problem of a spin wave travelling in an anti-ferromagnetic plate.

2 Spin wave propagation in the antiferromagnetic plate

Let us consider planar antiferromagnetic (AFM) containing two magnetic sublattices with magnetizations \vec{M}_1 and \vec{M}_2 . Total magnetization of the AFM $\vec{M} = \vec{M}_1 + \vec{M}_2$ in the ground state is equal zero ($\vec{M}_1 = -\vec{M}_2$, $|\vec{M}_1| = |\vec{M}_2| = M_0$).

We will use the σ -model approach based on the equation for the antiferromagnetism vector $\vec{l} = (\vec{M}_1 - \vec{M}_2)/2M_0$ [2, 3, 4]. The effective Lagrangian of the σ -model for the \vec{l} vector reads as [5]:

$$\mathcal{L} = \frac{\alpha M_0^2}{2} \int \left\{ \frac{1}{c^2} \left(\frac{\partial \vec{l}}{\partial t} \right)^2 - (\nabla \vec{l})^2 - w(\vec{l}) \right\} d^2x, \quad (1)$$

where $w(\vec{l}) = \frac{1}{2}\beta_1 l_y^2 + \frac{1}{2}\beta_2 l_z^2$ ($0 < \beta_1 < \beta_2$) is anisotropy energy, $c = \gamma M_0 \sqrt{\alpha \delta / 2}$ ¹. The phenomenological constants δ and α describe the homogeneous and inhomogeneous exchange interactions, respectively.

The dynamic equations for \vec{l} can be written as Euler-Lagrange equations for the Lagrangian (1). Using $\vec{l}^2 = 1$ these equations may be presented in the form [5]:

$$\left[\vec{l} \times \frac{\delta \mathcal{L}}{\delta \vec{l}} \right] = 0. \quad (2)$$

For the planar AFM it is convenient to represent the dynamics of unit vector \vec{l} by means of the angular variables:

$$l_3 = \cos \theta, \quad l_1 + i l_2 = \sin \theta \exp(i\varphi); \quad (3)$$

where the polar axis is directed along the easy axis of the AFM. The equations of motion for θ and φ can be written in the form:

$$\alpha \left(\nabla^2 \theta - \frac{1}{c^2} \frac{\partial^2 \theta}{\partial t^2} \right) + \alpha \sin \theta \cos \theta \left[(\nabla \varphi)^2 - \frac{1}{c^2} \left(\frac{\partial \varphi}{\partial t} \right)^2 \right] - \frac{\partial w_a}{\partial \theta} = 0,$$

¹In the simplest case of zero field and zero Dzyaloshinskii interaction.

$$\alpha \nabla (\sin^2 \theta \nabla \varphi) - \frac{\alpha}{c^2} \frac{\partial}{\partial t} \left(\sin^2 \theta \frac{\partial \varphi}{\partial t} \right) - \frac{\partial w_a}{\partial \varphi} = 0. \quad (4)$$

We will search the solution like travelling wave which propagate in the $X0Y$ plane ($\theta = \pi/2$) and $\varphi = \varphi((\vec{k}\vec{r} - \omega t)/k\sqrt{\alpha})$. In this the case equations (4) give:

$$\left(\frac{V^2}{c^2} - 1 \right) \frac{\partial^2 \varphi}{\partial \xi^2} + \sin \varphi \cos \varphi = 0, \quad (5)$$

where $V = \omega/k$, \vec{k} - wave vector and \vec{r} radius-vector in the magnetic plane, ω - frequency.

At small k phase velocity of spin wave increases infinitely that is why $V > c$, equation (5) has stable harmonic solution and Cherenkov radiation takes place.

Vectors \vec{l} and $\dot{\vec{l}}$ oscillate in the AFM plane, thus the magnetization vector $\vec{M} \sim [\vec{l} \times \dot{\vec{l}}]$ is normal to the AFM plane. Evidently, if $\varphi \ll 1$ the dependence of magnetization vector from time and space variable becomes:

$$\vec{M} = \vec{M}_0 \exp(-i\omega t + i\vec{k}\vec{r}), \quad (6)$$

where $\vec{M}_0 = (0, 0, M_0)$, ω - frequency and \vec{k} wave vector of the travelling wave.

In the following (next) section we will show that dependence (6) allows to receive the exact solution of the radiation problem.

3 Exact solution of the Cherenkov radiation problem of a spin wave

At a given functional dependence of magnet dipole moment \vec{M} on the space and time variable the exact solution of D'Alembertian equation for the magnetic Hertzian vector $\vec{\Pi}_m$:

$$\Delta \vec{\Pi}_m - \frac{1}{c^2} \frac{\partial^2 \vec{\Pi}_m}{\partial t^2} = -4\pi \vec{\Pi}_m, \quad (7)$$

is expressed by the retarded potential:

$$\vec{\Pi}_m = \frac{1}{c} \int_V \frac{\vec{M}(\vec{r}', t - \frac{|\vec{r} - \vec{r}'|}{c}) dV'}{|\vec{r} - \vec{r}'|}, \quad (8)$$

where \vec{M} is a magnetic dipole moment per area unit, \vec{r} is the distance between the origin and the observation point, \vec{r}' is the distance between the origin and the source point where the magnet dipole moment element is placed.

Carrying out the integration (8) one can find the vector potential \vec{A} and scalar potential φ via:

$$\vec{A} = \text{rot} \vec{\Pi}_m, \quad \varphi = 0. \quad (9)$$

And the expression of electric and magnetic fields as usual:

$$\begin{aligned} \vec{E} &= -\frac{1}{c} \frac{\partial}{\partial t} \text{rot} \vec{\Pi}_m \\ \vec{H} &= \text{rot rot} \vec{\Pi}_m. \end{aligned} \quad (10)$$

It is easy to test that Lorentz gauge is satisfied identically at the made definitions.

Taking into account that a magnet moment in (8) has to contain a retarded time, we shall obtain:

$$\begin{aligned} \vec{\Pi}_m &= \frac{\vec{M}_0}{c} \int_V \frac{\exp(-i\omega t + \frac{i\omega}{c} |\vec{r} - \vec{r}'| + ik_x x') dV'}{|\vec{r} - \vec{r}'|} \\ &= \frac{\vec{M}_0 \cdot \exp(-i\omega t + ik_x x)}{c} \cdot \\ &\quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp\left(\frac{i\omega}{c} \sqrt{(x - x')^2 + (y - y')^2 + z^2} + ik_x(x' - x)\right) dx' dy'}{\sqrt{(x - x')^2 + (y - y')^2 + z^2}} \end{aligned} \quad (11)$$

We assume z -axis to be directed along the normal to the plate and x -axis coincides with the wave vector \vec{k} . Evidently, that $\vec{k}\vec{r} = kx$.

Let's use the polar coordinates $x' - x = r \cos \varphi$ and $y' - y = r \sin \varphi$ for the further calculation. This gives:

$$\begin{aligned}
\vec{\Pi}_m &= \frac{\vec{M}_0 \exp(-i\omega t + ikx)}{c} \cdot \int_0^\infty \int_0^{2\pi} \frac{\exp(\frac{i\omega}{c} \sqrt{r^2 + z^2}) \exp(ikr \cos \varphi) r dr d\varphi}{\sqrt{r^2 + z^2}} \\
&= \frac{\vec{M}_0 \exp(-i\omega t + ikx)}{c} \int_0^\infty \int_0^{2\pi} \exp(ikr \cos \varphi) d\varphi \frac{\exp(\frac{i\omega}{c} \sqrt{r^2 + z^2}) r dr}{\sqrt{r^2 + z^2}} \\
&= 2\pi \frac{\vec{M}_0 \exp(-i\omega t + ikx)}{c} \int_0^\infty J_0(kr) \frac{\exp(\frac{i\omega}{c} \sqrt{r^2 + z^2}) r dr}{\sqrt{r^2 + z^2}}, \tag{12}
\end{aligned}$$

where $J_0(kr)$ is the Bessel function of the first kind of zero order. In (12) the well-known equality for Bessel functions was used (e.g. 9.1.21 in [6]).

Let's use the Euler's formula and represent the exponential form (12) in the trigonometric one (by sine and cosine):

$$\begin{aligned}
\vec{\Pi}_m &= 2\pi \frac{\vec{M}_0 \exp(-i\omega t + ikx)}{c} \int_0^\infty J_0(kr) \frac{\cos(\frac{\omega}{c} \sqrt{r^2 + z^2}) r dr}{\sqrt{r^2 + z^2}} \\
&\quad + 2\pi i \frac{\vec{M}_0 \exp(-i\omega t + ikx)}{c} \int_0^\infty J_0(kr) \frac{\sin(\frac{\omega}{c} \sqrt{r^2 + z^2}) r dr}{\sqrt{r^2 + z^2}} \tag{13}
\end{aligned}$$

Each integral from (13) can be evaluated exactly (not approximately), by virtue [7, p. 775, (6.737)]:

$$\int_0^\infty J_0(kr) \frac{\cos(\frac{\omega}{c} \sqrt{r^2 + z^2}) r dr}{\sqrt{r^2 + z^2}} = \begin{cases} -\sqrt{\frac{\pi z}{2}} \frac{N_{-\frac{1}{2}}(|z| \sqrt{\frac{\omega^2}{c^2} - k^2})}{\sqrt[4]{\frac{\omega^2}{c^2} - k^2}}, & |\frac{\omega}{c}| > k; \\ \sqrt{\frac{2z}{\pi}} \frac{K_{\frac{1}{2}}(|z| \sqrt{k^2 - \frac{\omega^2}{c^2}})}{\sqrt[4]{k^2 - \frac{\omega^2}{c^2}}}, & |\frac{\omega}{c}| < k. \end{cases} \tag{14}$$

$$\int_0^\infty J_0(kr) \frac{\sin(\frac{\omega}{c} \sqrt{r^2 + z^2}) r dr}{\sqrt{r^2 + z^2}} = \begin{cases} \sqrt{\frac{\pi z}{2}} \frac{J_{-\frac{1}{2}}(|z| \sqrt{\frac{\omega^2}{c^2} - k^2})}{\sqrt[4]{\frac{\omega^2}{c^2} - k^2}}, & |\frac{\omega}{c}| > k; \\ 0, & |\frac{\omega}{c}| < k. \end{cases} \tag{15}$$

Bessel function of half-integer order may be expressed through elementary functions ([6]):

$$\begin{aligned} N_{-\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}} \sin(x), & K_{\pm\frac{1}{2}}(x) &= \sqrt{\frac{\pi}{2x}} \exp(-x) \\ J_{-\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}} \cos(x), \end{aligned} \quad (16)$$

where $N_{-\frac{1}{2}}(x)$ and $K_{\pm\frac{1}{2}}(x)$ are modified Bessel's functions of half-integer order. It is convenient to make the following definition:

$$q = \begin{cases} \sqrt{\frac{\omega^2}{c^2} - k^2}, & |\frac{\omega}{c}| > k \\ i\sqrt{k^2 - \frac{\omega^2}{c^2}}, & |\frac{\omega}{c}| < k \end{cases} \quad (17)$$

One can return to the exponents and with respect to (16) express (13) in the form:

$$\vec{\Pi}_m = \frac{i}{q} \frac{2\pi \vec{M}_0}{c} \exp(-i\omega t + ikx + iq|z|) \quad (18)$$

Expression (18) describes typical case of Cherenkov radiation. If wave velocity exceeds the light one then the *TEM* wave is radiated. In the opposite case electromagnetic fields exponentially decrease in z -direction.

Using (10) we readily find the electric and magnetic fields expression:

$$\begin{aligned} \vec{E} &= \vec{e}_2 \frac{i}{q} \frac{2\pi \vec{M}_0 \omega k^2}{c} \exp(-i\omega t + ikx + iq|z|) \\ \vec{H} &= i \frac{k}{q} \frac{2\pi \vec{M}_0}{c} \exp(-i\omega t + ikx + iq|z|) \{ \mp q\vec{e}_1 + k\vec{e}_3 \}, \end{aligned} \quad (19)$$

where $\vec{e}_1, \vec{e}_2, \vec{e}_3$ - unit Cartesian vectors, signs \mp correspond to the $z > 0$ and $z < 0$ respectively.

Also easy the Pointing vector can be written:

$$\vec{S} = \frac{\pi \omega M_0^2 k^2}{q^2 c^2} \{ \pm q\vec{e}_3 + k\vec{e}_1 \}, \quad (20)$$

where signs \pm correspond to the $z > 0$ and $z < 0$ respectively.

4 Conclusion

Evidently, that the travelling wave of the electric dipoles at the plate like (6):

$$\vec{P} = \vec{P}_0 \exp(-i\omega t + i\vec{k}\vec{r}),$$

gives the same solution as (18) with the simple changes $\vec{\Pi}_m \rightarrow \vec{\Pi}_e$ and $\vec{M}_0 \rightarrow \vec{P}_0$ and corresponding expressions for fields:

$$\vec{A} = \frac{1}{c} \frac{\partial \vec{\Pi}_e}{\partial t}, \quad \varphi = -\operatorname{div} \vec{\Pi}_e$$

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